



0962CH08

CHAPTER 8

QUADRILATERALS

8.1 Properties of a Parallelogram

You have already studied quadrilaterals and their types in Class VIII. A quadrilateral has four sides, four angles and four vertices. A parallelogram is a quadrilateral in which both pairs of opposite sides are parallel.

Let us perform an activity.

Cut out a parallelogram from a sheet of paper and cut it along a diagonal (see Fig. 8.1). You obtain two triangles. What can you say about these triangles?

Place one triangle over the other. Turn one around, if necessary. What do you observe?

Observe that the two triangles are congruent to each other.

Repeat this activity with some more parallelograms. Each time you will observe that each diagonal divides the parallelogram into two congruent triangles.

Let us now prove this result.

Theorem 8.1 : *A diagonal of a parallelogram divides it into two congruent triangles.*

Proof : Let ABCD be a parallelogram and AC be a diagonal (see Fig. 8.2). Observe that the diagonal AC divides parallelogram ABCD into two triangles, namely, $\triangle ABC$ and $\triangle CDA$. We need to prove that these triangles are congruent.

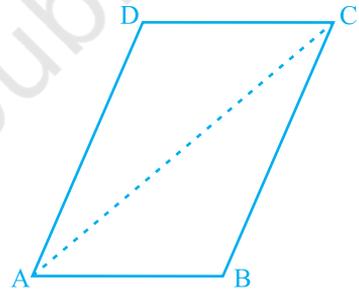


Fig. 8.1

In $\triangle ABC$ and $\triangle CDA$, note that $BC \parallel AD$ and AC is a transversal.

So, $\angle BCA = \angle DAC$ (Pair of alternate angles)

Also, $AB \parallel DC$ and AC is a transversal.

So, $\angle BAC = \angle DCA$ (Pair of alternate angles)

and $AC = CA$ (Common)

So, $\triangle ABC \cong \triangle CDA$ (ASA rule)

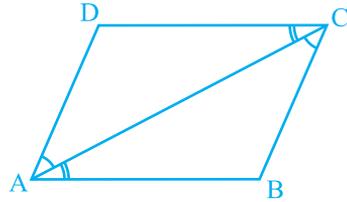


Fig. 8.2

or, diagonal AC divides parallelogram $ABCD$ into two congruent triangles ABC and CDA .

Now, measure the opposite sides of parallelogram $ABCD$. What do you observe?

You will find that $AB = DC$ and $AD = BC$.

This is another property of a parallelogram stated below:

Theorem 8.2 : *In a parallelogram, opposite sides are equal.*

You have already proved that a diagonal divides the parallelogram into two congruent triangles; so what can you say about the corresponding parts say, the corresponding sides? They are equal.

So, $AB = DC$ and $AD = BC$

Now what is the converse of this result? You already know that whatever is given in a theorem, the same is to be proved in the converse and whatever is proved in the theorem it is given in the converse. Thus, Theorem 8.2 can be stated as given below :

If a quadrilateral is a parallelogram, then each pair of its opposite sides is equal. So its converse is :

Theorem 8.3 : *If each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram.*

Can you reason out why?

Let sides AB and CD of the quadrilateral $ABCD$ be equal and also $AD = BC$ (see Fig. 8.3). Draw diagonal AC .

Clearly, $\triangle ABC \cong \triangle CDA$ (Why?)

So, $\angle BAC = \angle DCA$

and $\angle BCA = \angle DAC$ (Why?)

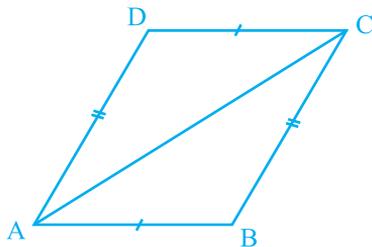


Fig. 8.3

Can you now say that $ABCD$ is a parallelogram? Why?

You have just seen that in a parallelogram each pair of opposite sides is equal and conversely if each pair of opposite sides of a quadrilateral is equal, then it is a parallelogram. Can we conclude the same result for the pairs of opposite angles?

Draw a parallelogram and measure its angles. What do you observe?

Each pair of opposite angles is equal.

Repeat this with some more parallelograms. We arrive at yet another result as given below.

Theorem 8.4 : *In a parallelogram, opposite angles are equal.*

Now, is the converse of this result also true? Yes. Using the angle sum property of a quadrilateral and the results of parallel lines intersected by a transversal, we can see that the converse is also true. So, we have the following theorem :

Theorem 8.5 : *If in a quadrilateral, each pair of opposite angles is equal, then it is a parallelogram.*

There is yet another property of a parallelogram. Let us study the same. Draw a parallelogram ABCD and draw both its diagonals intersecting at the point O (see Fig. 8.4).

Measure the lengths of OA, OB, OC and OD.

What do you observe? You will observe that

$$OA = OC \quad \text{and} \quad OB = OD.$$

or, O is the mid-point of both the diagonals.

Repeat this activity with some more parallelograms.

Each time you will find that O is the mid-point of both the diagonals.

So, we have the following theorem :

Theorem 8.6 : *The diagonals of a parallelogram bisect each other.*

Now, what would happen, if in a quadrilateral the diagonals bisect each other? Will it be a parallelogram? Indeed this is true.

This result is the converse of the result of Theorem 8.6. It is given below:

Theorem 8.7 : *If the diagonals of a quadrilateral bisect each other, then it is a parallelogram.*

You can reason out this result as follows:

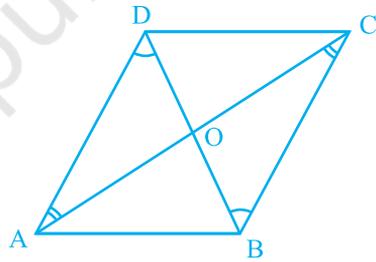


Fig. 8.4

Note that in Fig. 8.5, it is given that $OA = OC$ and $OB = OD$.

So, $\triangle AOB \cong \triangle COD$ (Why?)

Therefore, $\angle ABO = \angle CDO$ (Why?)

From this, we get $AB \parallel CD$

Similarly, $BC \parallel AD$

Therefore ABCD is a parallelogram.

Let us now take some examples.

Example 1 : Show that each angle of a rectangle is a right angle.

Solution : Let us recall what a rectangle is.

A rectangle is a parallelogram in which one angle is a right angle.

Let ABCD be a rectangle in which $\angle A = 90^\circ$.

We have to show that $\angle B = \angle C = \angle D = 90^\circ$

We have, $AD \parallel BC$ and AB is a transversal (see Fig. 8.6).

So, $\angle A + \angle B = 180^\circ$ (Interior angles on the same side of the transversal)

But, $\angle A = 90^\circ$

So, $\angle B = 180^\circ - \angle A = 180^\circ - 90^\circ = 90^\circ$

Now, $\angle C = \angle A$ and $\angle D = \angle B$

(Opposite angles of the parallelogram)

So, $\angle C = 90^\circ$ and $\angle D = 90^\circ$.

Therefore, each of the angles of a rectangle is a right angle.

Example 2 : Show that the diagonals of a rhombus are perpendicular to each other.

Solution : Consider the rhombus ABCD (see Fig. 8.7).

You know that $AB = BC = CD = DA$ (Why?)

Now, in $\triangle AOD$ and $\triangle COD$,

$OA = OC$ (Diagonals of a parallelogram bisect each other)

$OD = OD$ (Common)

$AD = CD$

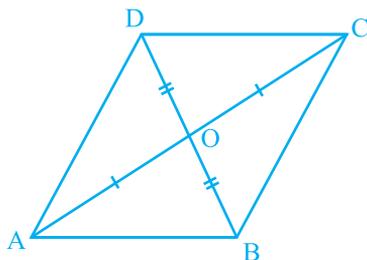


Fig. 8.5



Fig. 8.6

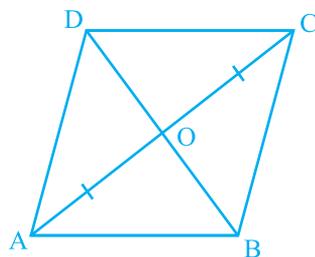


Fig. 8.7

Therefore, $\triangle AOD \cong \triangle COD$
(SSS congruence rule)

This gives, $\angle AOD = \angle COD$ (CPCT)

But, $\angle AOD + \angle COD = 180^\circ$ (Linear pair)

So, $2\angle AOD = 180^\circ$

or, $\angle AOD = 90^\circ$

So, the diagonals of a rhombus are perpendicular to each other.

Example 3 : ABC is an isosceles triangle in which $AB = AC$. AD bisects exterior angle PAC and $CD \parallel AB$ (see Fig. 8.8). Show that

(i) $\angle DAC = \angle BCA$ and (ii) ABCD is a parallelogram.

Solution : (i) $\triangle ABC$ is isosceles in which $AB = AC$ (Given)

So, $\angle ABC = \angle ACB$ (Angles opposite to equal sides)

Also, $\angle PAC = \angle ABC + \angle ACB$
(Exterior angle of a triangle)

or, $\angle PAC = 2\angle ACB$ (1)

Now, AD bisects $\angle PAC$.

So, $\angle PAC = 2\angle DAC$ (2)

Therefore,

$2\angle DAC = 2\angle ACB$ [From (1) and (2)]

or, $\angle DAC = \angle ACB$

(ii) Now, these equal angles form a pair of alternate angles when line segments BC and AD are intersected by a transversal AC.

So, $BC \parallel AD$

Also, $BA \parallel CD$ (Given)

Now, both pairs of opposite sides of quadrilateral ABCD are parallel.

So, ABCD is a parallelogram.

Example 4 : Two parallel lines l and m are intersected by a transversal p (see Fig. 8.9). Show that the quadrilateral formed by the bisectors of interior angles is a rectangle.

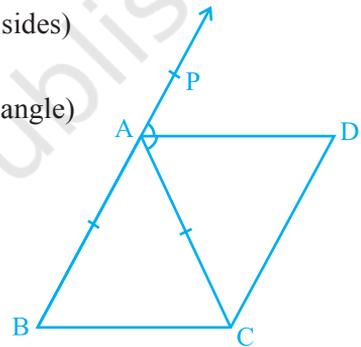


Fig. 8.8

Solution : It is given that $PS \parallel QR$ and transversal p intersects them at points A and C respectively.

The bisectors of $\angle PAC$ and $\angle ACQ$ intersect at B and bisectors of $\angle ACR$ and $\angle SAC$ intersect at D.

We are to show that quadrilateral ABCD is a rectangle.

Now, $\angle PAC = \angle ACR$

(Alternate angles as $l \parallel m$ and p is a transversal)

$$\text{So, } \frac{1}{2} \angle PAC = \frac{1}{2} \angle ACR$$

$$\text{i.e., } \angle BAC = \angle ACD$$

These form a pair of alternate angles for lines AB and DC with AC as transversal and they are equal also.

$$\text{So, } AB \parallel DC$$

$$\text{Similarly, } BC \parallel AD \quad (\text{Considering } \angle ACB \text{ and } \angle CAD)$$

Therefore, quadrilateral ABCD is a parallelogram.

$$\text{Also, } \angle PAC + \angle CAS = 180^\circ \quad (\text{Linear pair})$$

$$\text{So, } \frac{1}{2} \angle PAC + \frac{1}{2} \angle CAS = \frac{1}{2} \times 180^\circ = 90^\circ$$

$$\text{or, } \angle BAC + \angle CAD = 90^\circ$$

$$\text{or, } \angle BAD = 90^\circ$$

So, ABCD is a parallelogram in which one angle is 90° .

Therefore, ABCD is a rectangle.

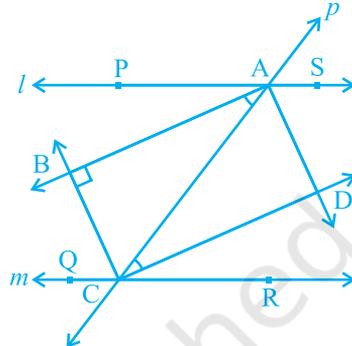


Fig. 8.9

Example 5 : Show that the bisectors of angles of a parallelogram form a rectangle.

Solution : Let P, Q, R and S be the points of intersection of the bisectors of $\angle A$ and $\angle B$, $\angle B$ and $\angle C$, $\angle C$ and $\angle D$, and $\angle D$ and $\angle A$ respectively of parallelogram ABCD (see Fig. 8.10).

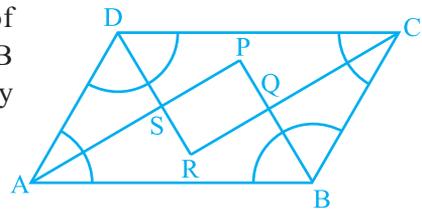


Fig. 8.10

In $\triangle ASD$, what do you observe?

Since DS bisects $\angle D$ and AS bisects $\angle A$, therefore,

$$\begin{aligned}
 \angle DAS + \angle ADS &= \frac{1}{2} \angle A + \frac{1}{2} \angle D \\
 &= \frac{1}{2} (\angle A + \angle D) \\
 &= \frac{1}{2} \times 180^\circ \quad (\angle A \text{ and } \angle D \text{ are interior angles} \\
 &\quad \text{on the same side of the transversal}) \\
 &= 90^\circ
 \end{aligned}$$

Also, $\angle DAS + \angle ADS + \angle DSA = 180^\circ$ (Angle sum property of a triangle)

or, $90^\circ + \angle DSA = 180^\circ$

or, $\angle DSA = 90^\circ$

So, $\angle PSR = 90^\circ$ (Being vertically opposite to $\angle DSA$)

Similarly, it can be shown that $\angle APB = 90^\circ$ or $\angle SPQ = 90^\circ$ (as it was shown for $\angle DSA$). Similarly, $\angle PQR = 90^\circ$ and $\angle SRQ = 90^\circ$.

So, PQRS is a quadrilateral in which all angles are right angles.

Can we conclude that it is a rectangle? Let us examine. We have shown that $\angle PSR = \angle PQR = 90^\circ$ and $\angle SPQ = \angle SRQ = 90^\circ$. So both pairs of opposite angles are equal.

Therefore, PQRS is a parallelogram in which one angle (in fact all angles) is 90° and so, PQRS is a rectangle.

EXERCISE 8.1

- If the diagonals of a parallelogram are equal, then show that it is a rectangle.
- Show that the diagonals of a square are equal and bisect each other at right angles.
- Diagonal AC of a parallelogram ABCD bisects $\angle A$ (see Fig. 8.11). Show that
 - it bisects $\angle C$ also,
 - ABCD is a rhombus.
- ABCD is a rectangle in which diagonal AC bisects $\angle A$ as well as $\angle C$. Show that: (i) ABCD is a square (ii) diagonal BD bisects $\angle B$ as well as $\angle D$.

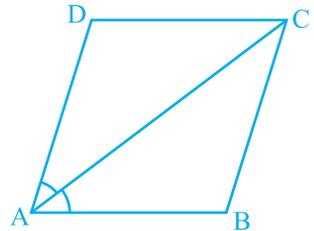


Fig. 8.11

5. In parallelogram ABCD, two points P and Q are taken on diagonal BD such that $DP = BQ$ (see Fig. 8.12). Show that:
- $\triangle APD \cong \triangle CQB$
 - $AP = CQ$
 - $\triangle AQB \cong \triangle CPD$
 - $AQ = CP$
 - APCQ is a parallelogram

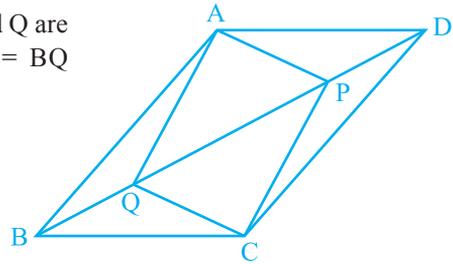


Fig. 8.12

6. ABCD is a parallelogram and AP and CQ are perpendiculars from vertices A and C on diagonal BD (see Fig. 8.13). Show that
- $\triangle APB \cong \triangle CQD$
 - $AP = CQ$

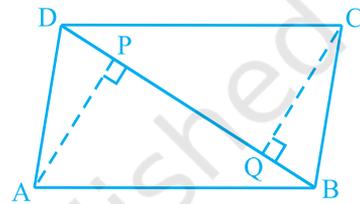


Fig. 8.13

7. ABCD is a trapezium in which $AB \parallel CD$ and $AD = BC$ (see Fig. 8.14). Show that
- $\angle A = \angle B$
 - $\angle C = \angle D$
 - $\triangle ABC \cong \triangle BAD$
 - diagonal $AC =$ diagonal BD

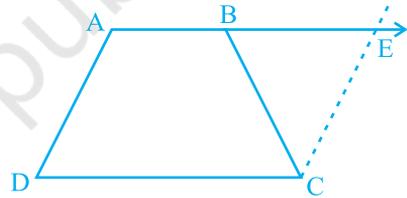


Fig. 8.14

[Hint : Extend AB and draw a line through C parallel to DA intersecting AB produced at E.]

8.2 The Mid-point Theorem

You have studied many properties of a triangle as well as a quadrilateral. Now let us study yet another result which is related to the mid-point of sides of a triangle. Perform the following activity.

Draw a triangle and mark the mid-points E and F of two sides of the triangle. Join the points E and F (see Fig. 8.15).

Measure EF and BC. Measure $\angle AEF$ and $\angle ABC$.

What do you observe? You will find that :

$$EF = \frac{1}{2} BC \text{ and } \angle AEF = \angle ABC$$

so, $EF \parallel BC$

Repeat this activity with some more triangles.

So, you arrive at the following theorem:

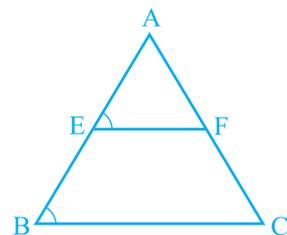


Fig. 8.15

Theorem 8.8 : *The line segment joining the mid-points of two sides of a triangle is parallel to the third side.*

You can prove this theorem using the following clue:

Observe Fig 8.16 in which E and F are mid-points of AB and AC respectively and $CD \parallel BA$.

$$\triangle AEF \cong \triangle CDF \quad (\text{ASA Rule})$$

So, $EF = DF$ and $BE = AE = DC$ (Why?)

Therefore, BCDE is a parallelogram. (Why?)

This gives $EF \parallel BC$.

In this case, also note that $EF = \frac{1}{2} ED = \frac{1}{2} BC$.

Can you state the converse of Theorem 8.8? Is the converse true?

You will see that converse of the above theorem is also true which is stated as below:

Theorem 8.9 : *The line drawn through the mid-point of one side of a triangle, parallel to another side bisects the third side.*

In Fig 8.17, observe that E is the mid-point of AB, line l is passing through E and is parallel to BC and $CM \parallel BA$.

Prove that $AF = CF$ by using the congruence of $\triangle AEF$ and $\triangle CDF$.

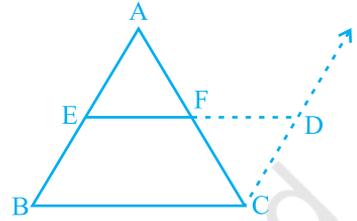


Fig. 8.16

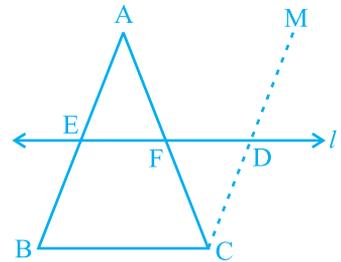


Fig. 8.17

Example 6 : In $\triangle ABC$, D, E and F are respectively the mid-points of sides AB, BC and CA (see Fig. 8.18). Show that $\triangle ABC$ is divided into four congruent triangles by joining D, E and F.

Solution : As D and E are mid-points of sides AB and BC of the triangle ABC, by Theorem 8.8,

$$DE \parallel AC$$

Similarly, $DF \parallel BC$ and $EF \parallel AB$

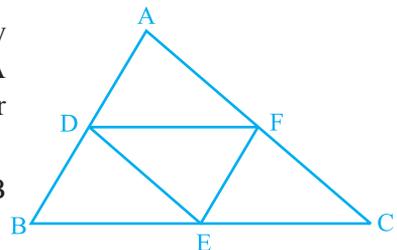


Fig. 8.18

Therefore ADEF, BDFE and DFCE are all parallelograms.

Now DE is a diagonal of the parallelogram BDFE,

therefore, $\triangle BDE \cong \triangle FED$

Similarly $\triangle DAF \cong \triangle FED$

and $\triangle EFC \cong \triangle FED$

So, all the four triangles are congruent.

Example 7 : l, m and n are three parallel lines intersected by transversals p and q such that l, m and n cut off equal intercepts AB and BC on p (see Fig. 8.19). Show that l, m and n cut off equal intercepts DE and EF on q also.

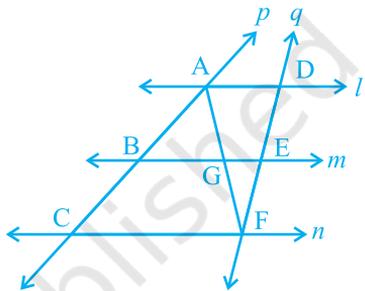


Fig. 8.19

Solution : We are given that $AB = BC$ and have to prove that $DE = EF$.

Let us join A to F intersecting m at G .

The trapezium $ACFD$ is divided into two triangles; namely $\triangle ACF$ and $\triangle AFD$.

In $\triangle ACF$, it is given that B is the mid-point of AC ($AB = BC$)

and $BG \parallel CF$ (since $m \parallel n$).

So, G is the mid-point of AF (by using Theorem 8.9)

Now, in $\triangle AFD$, we can apply the same argument as G is the mid-point of AF , $GE \parallel AD$ and so by Theorem 8.9, E is the mid-point of DF ,

i.e., $DE = EF$.

In other words, l, m and n cut off equal intercepts on q also.

EXERCISE 8.2

1. $ABCD$ is a quadrilateral in which P, Q, R and S are mid-points of the sides AB, BC, CD and DA (see Fig 8.20). AC is a diagonal. Show that :

- (i) $SR \parallel AC$ and $SR = \frac{1}{2} AC$
- (ii) $PQ = SR$
- (iii) $PQRS$ is a parallelogram.

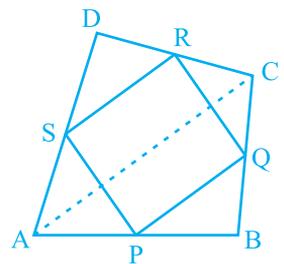


Fig. 8.20

2. ABCD is a rhombus and P, Q, R and S are the mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rectangle.
3. ABCD is a rectangle and P, Q, R and S are mid-points of the sides AB, BC, CD and DA respectively. Show that the quadrilateral PQRS is a rhombus.
4. ABCD is a trapezium in which $AB \parallel DC$, BD is a diagonal and E is the mid-point of AD. A line is drawn through E parallel to AB intersecting BC at F (see Fig. 8.21). Show that F is the mid-point of BC.

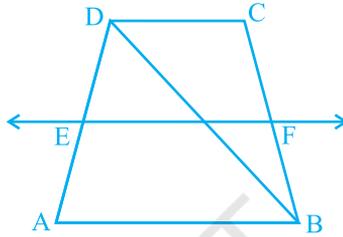


Fig. 8.21

5. In a parallelogram ABCD, E and F are the mid-points of sides AB and CD respectively (see Fig. 8.22). Show that the line segments AF and EC trisect the diagonal BD.

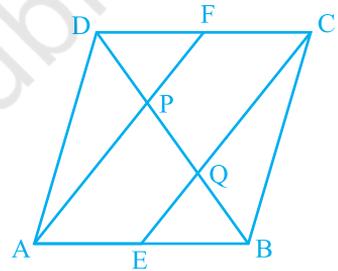


Fig. 8.22

6. ABC is a triangle right angled at C. A line through the mid-point M of hypotenuse AB and parallel to BC intersects AC at D. Show that
 - (i) D is the mid-point of AC
 - (ii) $MD \perp AC$
 - (iii) $CM = MA = \frac{1}{2} AB$

8.3 Summary

In this chapter, you have studied the following points :

1. A diagonal of a parallelogram divides it into two congruent triangles.
2. In a parallelogram,
 - (i) opposite sides are equal
 - (ii) opposite angles are equal
 - (iii) diagonals bisect each other
3. Diagonals of a rectangle bisect each other and are equal and vice-versa.
4. Diagonals of a rhombus bisect each other at right angles and vice-versa.
5. Diagonals of a square bisect each other at right angles and are equal, and vice-versa.
6. The line-segment joining the mid-points of any two sides of a triangle is parallel to the third side and is half of it.
7. A line through the mid-point of a side of a triangle parallel to another side bisects the third side.