Unit 3c

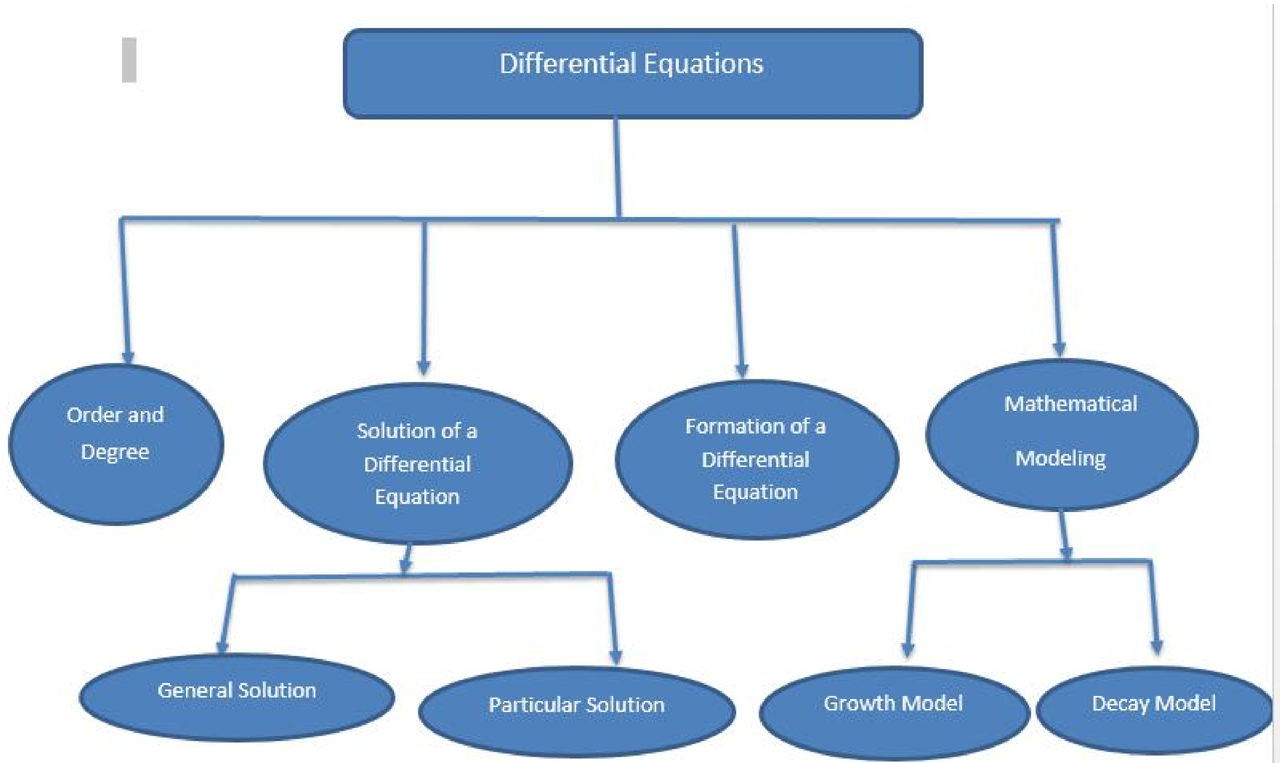
Differential Equations and Modeling

3.0 LEARNING OUTCOMES

After completion of this unit the students will be able to

- Determine order and degree of a differential equation.
- Solve the differential equation and understand the difference between general and particular solution
- Form differential equation by eliminating arbitrary constants.
- Formulate physical problem in terms of mathematical model and find its solution

Concept Map



3.10 Differential equations

For a given function, $y = f(x)$

$$\frac{dy}{dx} = g(x) \qquad \ldots(1) \quad \text{where } f'(x) = g(x)$$

An equation of the form (1) is known as differential equation.

These equations arise in various applications, may it be in Physics, Chemistry, Biology, Anthropology, Geology, Economics etc.

Hence an in-depth study of differential equation has assumed great importance in all the modern scientific investigations.

Let us give a formal definition of differential equation:

An equation involving derivative(s) of the dependent variable with respect to the independent variable(s) is called a differential equation.

A differential equation involving derivatives of the dependent variable with respect to only one independent variable is called an ordinary differential equation.

Some examples of ordinary differential equations are as follows

$$\frac{dy}{dx} + \frac{y}{x} = x^3 \qquad \ldots(1)$$

$$\frac{d^3y}{dx^3} + x^2\left(\frac{d^2y}{dx^2}\right)^3 = 0 \qquad \ldots(2)$$

$$xy\,\frac{d^2y}{dx^2} + x\left(\frac{dy}{dx}\right)^2 - y\,\frac{dy}{dx} = 0 \qquad \ldots(3)$$

$$\frac{d^2y}{dx^2} + y^2 + e^{\frac{dy}{dx}} = 0 \qquad \ldots(4)$$

$$\left(\frac{d^3y}{dx^3}\right)^2 - 3\frac{d^2y}{dx^2} + 2\left(\frac{dy}{dx}\right)^4 = y^4 \qquad \ldots.(5)$$

3.10.1 Order of a Differential Equation

The order of differential equation is defined as the highest ordered derivative of the dependent variable with respect to the independent variable involved in the differential equation.

The differential equations (1), (2), (3), (4) and (5) mentioned earlier involve the highest derivative of first, third, second, second and third order respectively. Therefore the order of these differential equations is 1, 3, 2, 2 and 3 respectively.

3.10.2 Degree of a Differential Equation

For the degree of a differential equation to be defined it must be a polynomial equation in its derivatives i.e., $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ etc.

The degree of a differential equation, when it is a polynomial equation in its derivatives is the highest power (positive integral index) of the highest order derivative involved in the differential equation. We observe that differential equations (1), (2), (3) and (5) are polynomial equations in its derivatives therefore their degrees are defined. But equation (4) is not a polynomial equation in $\frac{dy}{dx}$, therefore its degree is not defined.

In view of the above definition, the differential equation (1), (2), (3) and (5) have degrees 1, 1, 1 and 2 respectively.

Example 1

Find the order and degree (if defined) of the following differential equations:

(i) $\frac{dy}{dx} = ky$, where k is a scalar

(ii) $\left(\frac{ds}{dt}\right)^4 + 2s\,\frac{d^2s}{dt^2} = 0$

(iii) $\left(1 + \frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$

(iv) $y\,dx + x\log\left(\frac{y}{x}\right)\,dy - 2x\,dy = 0$

Solutions:

(i) $\frac{dy}{dx} = ky$

The highest order derivative present is $\frac{dy}{dx}$ and it is raised to power 1. So its order is 1 and degree is also 1.

(ii) $\left(\frac{ds}{dt}\right)^4 + 2\,s\,\frac{d^2s}{dt^2} = 0$

The highest order derivative present is $\frac{d^2s}{dt^2}$ and it is raised to power 1. So its order is 2 and degree is 1.

(iii) $\left(1 + \frac{dy}{dx}\right)^3 = \left(\frac{d^2y}{dx^2}\right)^2$

The highest order derivative present is $\frac{d^2y}{dX^2}$ and it is raised to power 2. So its order is 2 and degree is also 2.

(iv) $y\,dx + x\log\left(\frac{y}{x}\right)dy - 2xdy = 0$

$\Rightarrow \left[2x - x\log\left(\frac{y}{x}\right)\right]dy = y\,dx$

$$\Rightarrow \frac{dy}{dx} = \frac{y}{2x - x\log\left(\frac{y}{x}\right)}$$

The highest order derivative present is $\frac{dy}{dx}$ and it is raised to power 1. So its order is 1 and degree is also 1.

Exercise 1

Determine the order and degree (if defined) of the following differential equations:

1. $x \frac{dy}{dx} + 2y = x^2, \quad x \neq 0$

2. $\frac{dy}{dx} + e^y = 0$

3. $\frac{d^2y}{dx^2} + \frac{dy}{dx} - 6y = 0$

4. $\left(\frac{dy}{dx}\right)^4 + 3y\left(\frac{d^2y}{dx^2}\right) = 0$

5. $(y''')^2 + (y'')^3 + (y')^4 + y^5 = 0$, where $y' = \frac{dy}{dx}, \quad y'' = \frac{d^2y}{dx^2}$ *and* $y''' = \frac{d^3y}{dx^3}$

3.10.3 General and particular solutions of a Differential Equation

In earlier classes, we have done questions based on finding the solutions of the equations of the types:

$$x^2 + x - 2 = 0 \qquad ...(1)$$

$$\cos x + \sin 2x = 0 \qquad ...(2)$$

Solution of these equations is real numbers that satisfy the given equations. On substituting these numbers for the unknown x, the two sides of the equation becomes equal.

Now consider the differential equation, $\frac{dy}{dx} = \frac{1}{x}, \quad x > 0$. The function $y = log x$ is a solution of this differential equation. It is in fact a particular solution.

Let us consider a general form of this solution

i.e., $y = a + log\ x$, where a is an arbitrary constant

$$\Rightarrow \frac{dy}{dx} = \frac{1}{x}$$

The function $y = a + log x$, consists of one arbitrary constant (parameter) a and is called the general solution of the given differential equation.

Let us consider another function

$y = \phi(x) = ae^{2x} + be^{-x}$, *where* $a, b \in \mathbb{R}$ and are called arbitrary constants.

On differentiating, we get

$$\frac{dy}{dx} = 2ae^{2x} - be^{-x} \,\ldots\ldots\ldots\ (1)$$

$$\frac{d^2y}{dx^2} = 4ae^{2x} + be^{-x} \,\ldots\ldots\ (2)$$

(2) - (1) gives $\quad \frac{d^2y}{dx^2} - \frac{dy}{dx} = 2ae^{2x} + 2be^{-x}$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y$$

The curve $y = \phi(x)$ is called the solution curve (integral curve) of the differential equation so obtained.

If $a \text{ and } b$ are given some particular values say, $a = 2 \text{ and } b = -3$, then

$y = 2e^{2x} - 3e^{-x}$

$$\Rightarrow \frac{dy}{dx} = 4e^{2x} + 3e^{-x} \qquad \dots (3)$$

$$\Rightarrow \frac{d^2y}{dx^2} = 8e^{2x} - 3e^{-x} \qquad \dots (4)$$

(4) - (3) $\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = 4e^{2x} - 6e^{-x}$

$$\Rightarrow \frac{d^2y}{dx^2} - \frac{dy}{dx} = 2y$$

Since, $y = 2e^{2x} - 3e^{-x}$ does not contain any arbitrary constant therefore it is the particular solutions of the differential equation. We conclude by giving the formal definitions of the solutions of the differential equation.

3.10.4 General Solution

The function involving the variables and independent arbitrary constants is called the general solution of the differential equation.

A general solution of the differential equation in two variables x and y and having two arbitrary constants c_1, c_2 is of the form $f(x, y, c_1, c_2) = 0$

3.10.5 Particular Solution

A solution obtained from the general solution by giving particular values to arbitrary constants is called a particular solution of the differential equation. A particular solution in two variable $x \text{ and } y$ does not contain any arbitrary constant and is of the form,

$f(x, y) = 0$

Example 2

Verify that the function $y = ae^{bx}$ is a solution of the differential equation

$$\frac{d^2y}{dx^2} - b^2y = 0$$

Solution : Given function is

$$y = ae^{bx} \qquad \dots (1)$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = a\ b\ e^{bx} \qquad \dots (2)$$

Differentiating again with respect to x, we get

$$\frac{d^2y}{dx^2} = ab^2e^{bx} \quad \ldots (3)$$

Substituting the value of y and $\frac{d^2y}{dx^2}$ from (1) and (3) in the differential equation, we get

$$LHS = \frac{d^2y}{dx^2} - b^2y$$

$$= ab^2e^{bx} - b^2ae^{bx}$$

$$= 0 = RHS$$

Therefore, the given function is a solution of the differential equation.

Example 3

Verify that $y = ce^{-x^3}$ is the solution of the differential equation $\frac{dy}{dx} + 3x^2y = 0$. Also determine the solution curve of the given differential equation that passes through the point (0, 5)

Solution: We have,

$$y = ce^{-x^3} \quad \ldots (1)$$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = ce^{-x^3}(-3x^2) \quad \ldots (2)$$

Substituting value of y and $\frac{dy}{dx}$ from (1) and (2) in the differential equation, we get

$$LHS = \frac{dy}{dx} + 3x^2y = -3x^2ce^{-x^3} + 3x^2ce^{-x^3} = 0 = RHS$$

Therefore the function $y = ce^{-x^3}$ is a solution of the given differential equation.

To determine the solution curve passing through the point (0, 5) we find value of c first

$$5 = ce^0 \Rightarrow c = 5$$

Hence, $y = 5e^{-x^3}$ is the equation of the solution curve.

Example 4

Verify that $y = \frac{1}{x} - \log x$ is a solution of the differential equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} - y = \log x$$

Solution: We have, $y = \frac{1}{x} - \log x \quad \ldots (1)$

Differentiating both sides with respect to x, we get

$$\frac{dy}{dx} = -\frac{1}{x^2} - \frac{1}{x} \quad \ldots (2)$$

Differentiating again both sides with respect to x, we get

$$\frac{d^2y}{dx^2} = \frac{2}{x^3} + \frac{1}{x^2} \qquad \ldots(3)$$

Substituting value of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ from (1), (2) and (3), we get:

$LHS \quad = x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - y$

$= x^2\left(\frac{2}{x^3} + \frac{1}{x^2}\right) + x\left(-\frac{1}{x^2} - \frac{1}{x}\right) - \left(\frac{1}{x} - \log x\right)$

$= \frac{2}{x} + 1 - \frac{1}{x} - 1 - \frac{1}{x} + \log x$

$= \log x = RHS$

Therefore, the given function is a solution of the differential equation.

Example 5

Show that $y^2 = 4ax$ is a solution of the differential equation, $y = x\frac{dy}{dx} + a\frac{dx}{dy}$

Solution: We have, $y^2 = 4ax \qquad \ldots(1)$

Differentiating both sides with respect to x, we get

$$2y\frac{dy}{dx} = 4a$$

$$\Rightarrow \frac{dy}{dx} = \frac{2a}{y} \Rightarrow x\frac{dy}{dx} = \frac{2ax}{y} \qquad \ldots(2)$$

Also, $\frac{dx}{dy} = \frac{y}{2a} \Rightarrow a\frac{dx}{dy} = \frac{y}{2} \qquad \ldots(3)$

Substituting value of $x\frac{dy}{dx}$ and $a\frac{dx}{dy}$ from (2) and (3) in the differential equation, we get

$RHS \quad = x\frac{dy}{dx} + a\frac{dx}{dy}$

$= \frac{2ax}{y} + \frac{y}{2} = \frac{4ax+y^2}{2y}$

$= \frac{2y^2}{2y} = y = LHS$

Hence, $y^2 = 4ax$ is the solution of the given differential equation.

Exercise 2

Verify that given function (explicit or implicit) is a solution of the corresponding differential equation (Q1 to 6)

1. $y = ae^{-x}$: $\frac{dy}{dx} + y = 0$

2. $y = \sqrt{1+x^2}$: $\frac{dy}{dx} = \frac{xy}{1+x^2}$

3. $xy = \log y + c$: $\frac{dy}{dx} = \frac{y^2}{1-xy}$, $(xy \neq 1)$

4. $ax^2 + by^2 = 1$: $x(yy_2 + y_1^2) = yy_1$

   where $y_1 = \frac{dy}{dx}$, $y_2 = \frac{d^2y}{dx^2}$

5. $y = (a+bx)e^{2x}$: $y_2 - 4y_1 + 4y = 0$

6. $x^2 = 2y^2 \log y$: $(x^2 + y^2)\frac{dy}{dx} - xy = 0$

7. Verify that the function, $y = ke^x - 1$ is a solution of the differential equation $\frac{dy}{dx} = y + 1$. Also determine the value of the constant k so that the solution curve of the given differential equation passes through the point (0,1).

3.11. Formation of a Differential Equation

The equation of a circle having center at (2, 3) and radius 5 units is

$$(x-2)^2 + (y-3)^2 = 5^2 \qquad \dots(1)$$

Differentiating (1) with respect to x, we get

$$2(x-2) + 2(y-3)\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{2-x}{y-3}, \quad (y \neq 3)$$

Which is a differential equation representing the family of circles whose one member is a circle represented by equation (1)

Now let us consider the equation

$$y = mx + c \qquad \dots(2),$$

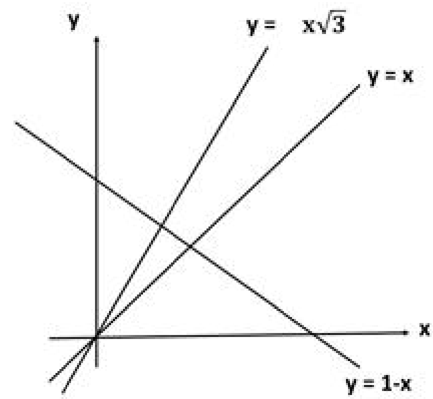
Where m is the slope and c is the y-intercept.

By giving different values to the parameters m and c, we get different members of the family as shown below:

$$y = x \qquad (m = 1,\ c = 0)$$

$$y = \sqrt{3}\,x \qquad (m = \sqrt{3},\ c = 0)$$

$$y = 1 - x \qquad (m = -1,\ c = 1)$$



We are interested in finding a differential equation that is satisfied by each member of the family. Further the equation must be free of m and c.

We have, $y = mx + c$

Differentiating both sides with respect to x, we get $\frac{dy}{dx} = m$

On differentiating again, we get

$$\frac{d^2y}{dx^2} = 0 \qquad \dots (3)$$

The differential equation (3) represents the family of straight lines given by equation (2)

Note: *Equation (2) is the general solution of the differential equation (3) and the differential equation is independent of the arbitrary constants.*

3.11.1 Steps to form a Differential Equation

Let us assume the given family of curves f depends on the parameters a, b (say) then it is represented by an equation of the form:

$$f(x, y, a, b) = 0 \qquad \dots(1)$$

Differentiating equation (1) with respect to x, we get

$$g(x, y, y', a, b) = 0 \qquad \dots(2)$$

But it is not possible to eliminate two parameters $a \text{ and } b$ from two equations. So a third equation is obtained by differentiating equation (2) with respect to x to obtain a relation of the form:

$$h(x, y, y', y'', a, b) = 0 \ \dots(3)$$

The required differential equation is obtained by eliminating $a \text{ and } b$ from equations (1), (2) and (3) to get

$$F(x, y, y', y'') = 0$$

Note: *If the given family of curves has n parameters then it is to be differentiated n times to eliminate the parameters and obtain the n^{th} order differential equation.*

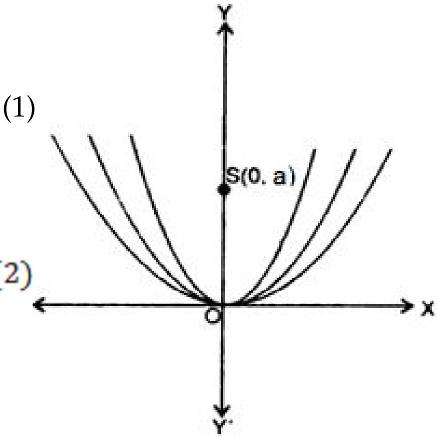
Example 6

Form a differential equation representing the family of parabolas having vertex at origin and axis along positive direction of y-axis.

Solution:

Equation of family of such parabolas is

$$x^2 = 4ay \quad \ldots (1)$$



Where a is the parameter

Differentiating (1) with respect to x , we get

$$2x = 4a\frac{dy}{dx} \qquad \Rightarrow 4a = \frac{2x}{\frac{dy}{dx}} \quad \ldots (2)$$

Eliminating a from equations (1) and (2), we get

$$x^2 = \frac{2x}{\frac{dy}{dx}}\, y$$

$$\Rightarrow x\frac{dy}{dx} = 2y$$

Example 7

Form a differential equation representing the family of curves given by

$y = ae^{bx}$, where a, b are arbitrary constants

Solution: We have, $y = ae^{bx}$ ... (1)

Differentiating both sides of (1) with respect to x, we get

$$\frac{dy}{dx} = abe^{bx}$$

$$\frac{dy}{dx} = by \quad (\because ae^{bx} = y) \Rightarrow b = \frac{1}{y}\frac{dy}{dx} \quad \ldots (2)$$

Differentiating again, we get

$$\frac{d^2y}{dx^2} = b\frac{dy}{dx} \qquad \Rightarrow \frac{d^2y}{dx^2} = \frac{1}{y}\left(\frac{dy}{dx}\right)^2 \qquad [using\ (2)]$$

Example 8

Form the differential equation of the family of hyperbolas having foci on x-axis and Centre at origin.

Solution: The equation of the family of hyperbolas having foci on x-axis and centre at origin is

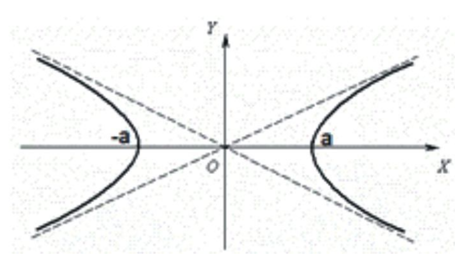
$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1 \quad \ldots (1)$$

Differentiating both sides of (1) with respect to x, we get

$$\frac{1}{a^2}2x - \frac{1}{b^2}\ 2y\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{x}{a^2} = \frac{y}{b^2}\frac{dy}{dx} \quad \Rightarrow \frac{y}{x}\frac{dy}{dx} = \frac{b^2}{a^2}$$

i.e., $\dfrac{yy'}{x} = \dfrac{b^2}{a^2} \quad \ldots (2) \qquad \left(\because \dfrac{dy}{dx} = y'\right)$



Differentiating both sides of (2) with respect to x, we get

$$\frac{x[y\,y''+(y')^2]-y\,y'}{x^2}=0$$

$$\Rightarrow xy\,y''+x(y')^2-y\,y'=0$$

Exercise 3

1. Form the differential equation not containing the arbitrary constants and satisfied by the equation $x^2-y^2=a^2$, where a is an arbitrary constant.
2. Find the differential equation of the family of circles having centre at origin.
3. Form the differential equation of the family of circles having centre on $y-axis$ and passing through origin.
4. Form the differential equation representing the family of curves $y=e^{2x}(a+bx)$, where a, b are arbitrary constants.
5. Find the differential equation representing the parabolas having their vertices at origin and foci on positive direction of *x*-axis.
6. Form the differential equation of the family of ellipses having their foci on $x-axis$ and centre at the origin.

3.11.2 Solving simple differential equation

In this section we shall solve some simple ordinary differential equations of first order and first degree

Case 1: $\frac{dy}{dx}=F(x)$ $\quad\quad\quad\quad \ldots.(i)$

The differential equation (i) can be expressed as $dy = F(x)dx$

Integrating both sides, we get

$$\int 1\,dy=\int F(x)dx$$

$\Rightarrow y=\int F(x)dx+c$, where c is an arbitrary constant

Case 2: $\frac{dy}{dx}=G(y)$ $\quad\quad \ldots(i)$

The differential equation (i) can be expressed as $\dfrac{dy}{G(y)}=dx$

Integrating both sides, we get

$$\int\frac{dy}{G(y)}=\int 1\,dx$$

$\Rightarrow x=\int\frac{dy}{G(y)}+c$, where c is arbitrary constant

Case 3: $\frac{dy}{dx} = F(x, y)$ ... (i)

If the differential equation can be expressed in the form:

$\frac{dy}{dx} = \emptyset(x)\,\psi(y)$, then $\frac{dy}{\psi(y)} = \emptyset(x)dx$

Integrating both sides, we get

$\int \frac{dy}{\psi(y)} = \int \emptyset(x)dx + c$, where c is an arbitrary constant

Example 9

Find the general solution of the differential equation $\frac{dy}{dx} + y = 1$, $(y \neq 1)$

Solution: $\frac{dy}{dx} = 1 - y$

$$\Rightarrow \int \frac{dy}{y-1} = -\int 1\, dx$$

$$\Rightarrow \log|y-1| = -x + c$$

$$\Rightarrow y - 1 = e^{c-x}$$

$$\Rightarrow y = ke^{-x} + 1, \qquad where\ e^c = k$$

Example 10

Solve the differential equation:

$$y \log y\, dx - x\, dy = 0$$

Solution: $y \log y\, dx = x\, dy$

$$\Rightarrow \int \frac{dx}{x} = \int \frac{dy}{y \log y}$$

$$\Rightarrow \log\,|x| + \log c = \log|log\ y|$$

$$\Rightarrow cx = \log y \ \Rightarrow y = e^{cx}$$

Example 11

Solve the differential equation:

$$\frac{dy}{dx} = e^{x+y} + x^2e^y$$

Solution: $\frac{dy}{dx} = e^x.e^y + x^2e^y$

$$\Rightarrow \frac{dy}{dx} = e^y(e^x + x^2)$$

$$\Rightarrow \frac{dy}{e^y} = (e^x + x^2)dx$$

$$\Rightarrow \int e^{-y}\,dy = \int (e^x + x^2)dx$$

$$\Rightarrow -e^{-y} = e^x + \frac{x^3}{3} + c$$

$$\Rightarrow e^x + e^{-y} + \frac{x^3}{3} = c$$

Example 12

Find the particular solution of the differential equation $\frac{dy}{dx} = x(2\log x + 1)$, given that $y = 0$ when $x = 2$

Solution: $dy = (2\,x log\ x + x)dx$

On integrating both sides, we get

$$\int 1\,dy = 2\int x\log x\,dx + \frac{x^2}{2} + c$$

$$\Rightarrow y = 2\left[log x\left(\frac{x^2}{2}\right) - \int \frac{1}{x}\frac{x^2}{2}\,dx\right] + \frac{x^2}{2} + c$$

$$\Rightarrow y = x^2 log\ x - \frac{x^2}{2} + \frac{x^2}{2} + c$$

$$\Rightarrow y = x^2 \log x + c \qquad \dots(1)$$

Since $y = 0$, when $x = 2$ $\quad\therefore 0 = 4\log 2 + c$

$$\Rightarrow c = -4 log 2$$

Substituting, $c = -4\log 2\ in(1)$, we get

$$y = x^2 log x - 4\log 2$$

Exercise 4

Find the general solution of the following differential equations given in Q.1 to 5

1. $\frac{dy}{dx} = (e^x + 1)y$
2. $x^5\frac{dy}{dx} = -y^5$
3. $\frac{dy}{dx} = \frac{x+1}{2-y}$
4. $x(e^{2y} - 1)dy + (x^2 - 1)e^y dx = 0$
5. $e^x\sqrt{1 - y^2}\,dx + \frac{y}{x}dy = 0$
6. Find the equation of the curve passing through the point (1, -1) whose differential equation is $xy\frac{dy}{dx} = (x + 2)(y + 2)$

7. Solve $(x+1)\frac{dy}{dx} = 2xy$, given that $y(2) = 3$

8. Find the particular solution of the differential equation $log\left(\frac{dy}{dx}\right) = 3x + 4y$, given that $y = 0$, when $x = 0$.

3.12 Differential Equations and Mathematical Modeling

Introduction

Differential equations play a pivotal role in modern world ranging from, engineering to ecology and from economics to biology.

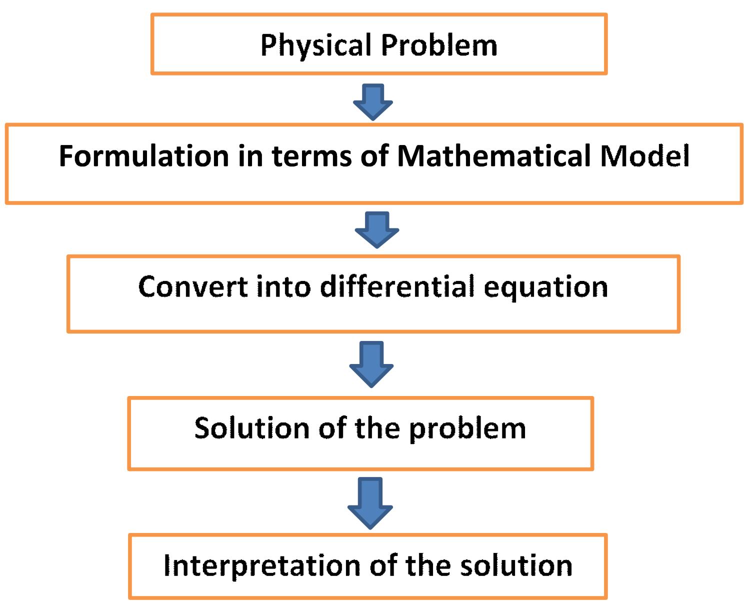
Algebra is sufficient to solve many static problems, but the most interesting natural phenomenon involve change and are described by equation that relate changing quantities known as differential equation. Many real-world problems involve rate of change of a quantity. These problems can be described by mathematical equations i.e., by mathematical models. Such, models have use in diverse range of applications such as astronomy, medicine, social science, financial mathematics etc. these mathematical models are examples of differential equations.

3.12.1 The process of Mathematical Modeling

In many natural phenomenon and real life applications a quantity changes at a rate proportional to the amount present. In such cases the amount present at time t is a function of t. The steps to be followed while solving mathematical modeling are as follows:

1. The formulation of a real-world problem in mathematical terms, i.e., the construction of a mathematical model.
2. The solution of the mathematical problem.
3. The interpretation of the mathematical result in the context of the original problem.

Mathematical Model of Physical Problem



3.12.2 Growth and Decay Models

The mathematical model for exponential growth or decay is given by

$$f(t) = A\,e^{kt} \qquad \text{or} \qquad y = Ae^{kt}$$

Where: t represents time

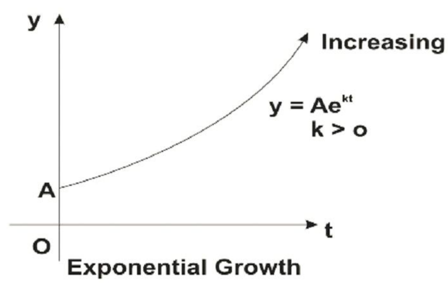
A the original amount

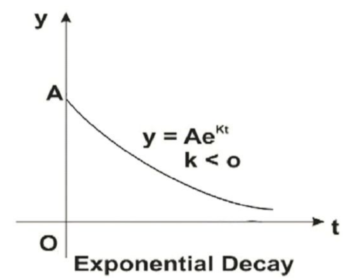
y or $f(t)$ represents the quantity at time t

k is a constant that depends on the rate of growth or decay

If $k > 0$, the formula represents exponential growth

If $k < 0$, the formula represents exponential decay





3.12.3 Population Growth

Suppose that $P(t)$ is the number of individuals in a population (of humans or insects or bacteria) having constant birth rate α and constant death rate β

Then the rate of change of population $P(t)$ with respect to time is given by

$$\frac{dP}{dt} = (\alpha - \beta)\,P \;\Rightarrow \frac{dP}{dt} = k\,P, \text{ Where } \alpha - \beta = k$$

$$\Rightarrow \int \frac{dP}{P} = \int k\,dt$$

$$\Rightarrow \log P = kt + c$$

$$\Rightarrow P = e^{kt+c}$$

$$\Rightarrow P = e^{c}\,e^{kt}$$

$$\Rightarrow P = A\,e^{kt} \text{ (Where } e^{c} = A)$$

$$\therefore P(t) = A\,e^{kt}\text{ , for all real } t$$

Example.13

In a certain culture of bacteria the rate of increase is proportional to the number present. It is found that there are 10,000 bacteria at the end of 3 hours and 40,000 bacteria at the end of 5 hours. How many bacteria were present in the beginning?

Solution: Let P be the number of bacteria after t hours

$$\frac{dP}{dt} \alpha\, P$$

$$\Rightarrow \frac{dP}{dt} = k\,P$$

$$\Rightarrow \int \frac{dP}{P} = \int k\,dt$$

$$\Rightarrow \log P = k\,t + c$$

$$\Rightarrow P = e^{kt+c} = e^{c} e^{kt}$$

$$\Rightarrow P(t) = \lambda\, e^{kt} \qquad \ldots(1) \qquad (\text{where } \; e^{c} = \lambda)$$

$$\because P(3) = 10{,}000 \qquad \therefore \lambda\, e^{3k} = 10{,}000 \quad \ldots(2)$$

$$P(5) = 40{,}000 \qquad \therefore \lambda e^{5k} = 40{,}000 \quad \ldots(3)$$

Dividing (3) by (2), we get

$$e^{2k} = 4 \quad \Rightarrow e^{k} = 2$$

Substituting, $e^{k} = 2$ in equation (2) we get

$$\lambda(2)^3 = 10{,}000 \Rightarrow \lambda = \frac{10000}{8} = 1250$$

From (1) we get, $P(t) = 1250\, e^{kt}$

Now, $P(0) = 1250e^{0} = 1250$

Hence we can say, there were 1250 bacteria in the beginning.

3.12.4 Compound Interest

A person deposits an amount $A(t)$ at a time t (in years) in a bank and suppose that the interest is compounded continuously at an annual interest rate r

To obtain the differential equation that governs the variation in the amount of money A in the bank with time t, we follow these steps:

During a short time interval $\Delta\, t$, the amount of interest added to the account is approximately given by

$$\Delta A = r \;\; A(t) \;\; \Delta t$$

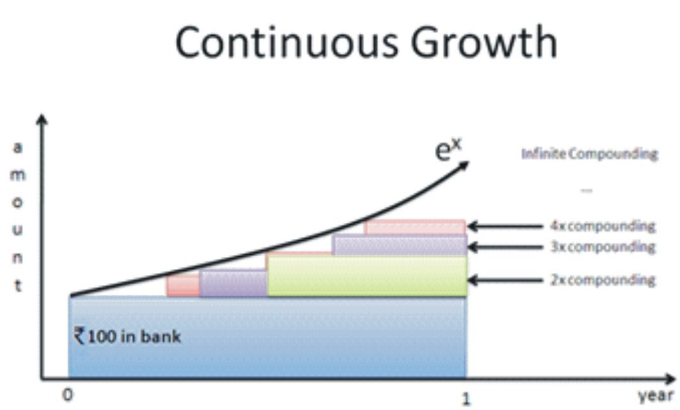
$$\therefore \frac{\Delta A}{\Delta t} = r\,A$$

$$\lim_{\Delta t \to o} \frac{\Delta A}{\Delta t} = rA \Rightarrow \frac{dA}{dt} = r\,A$$

$$\therefore \int \frac{dA}{A} = \int r\,dt$$

$$\Rightarrow \log A = \;\; rt + c$$

$$\Rightarrow A = e^{rt+c}$$



$$\Rightarrow A = e^c\, e^{rt}$$

$$\Rightarrow A = A_0\, e^{rt} \quad , \quad (e^c = A_o)$$

Where A_o is the money deposited at $t = 0$

Example 14

Ms. Rajni deposited Rs.10,000 in a bank that pays 4% interest compounded continuously .

a) How much amount will she get after 10 years?

b) How long it will take the money to double?

Solution: We know, $\frac{dA}{dt} = rA \Rightarrow \int \frac{dA}{A} = \int r\, dt$

$$\Rightarrow \log A = rt + c \;\Rightarrow A = e^{rt+c}$$

$$\Rightarrow A = A_o\, e^{rt} \qquad \dots(1)$$

At $t = 0,\; A = 10{,}000$

So (1) $\Rightarrow$ $10{,}000 = A_0$

Hence $A = 10{,}000\; e^{0.04\,t}$ $\quad\dots(2)$

a) $A(10) = 10{,}000\; e^{0.04 \times 10} = 10{,}000\; e^{0.4}$ (using (2))

$A(10) = 10{,}000 \times 1.49182 =$ ₹ 14918.2

b) We have to find t in which $A_0 = 10{,}000$ becomes

A=20,000. Using … (2)

$20{,}000 = 10{,}000\; e^{0.04t}$

$\Rightarrow e^{0.04t} = 2$

$\Rightarrow \log e^{0.04t} = \log_e 2$

$\Rightarrow 0.04t = \log_e 2 \Rightarrow t = \frac{\log_e 2}{0.04} = \frac{0.6931}{0.04}$

$t = 17.32\; years(approx)$

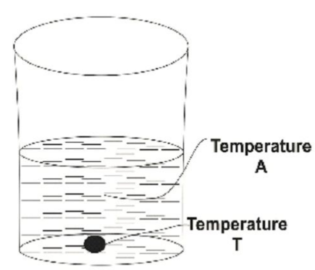
3.12.5 Newton's Law of Cooling

The rate of change of temperature T of a body is proportional to the difference between T and the temperature of the surrounding medium A.

$\frac{dT}{dt} = -k(T - A)$, where $k > 0$ constant

If $T > A$, then $\frac{dT}{dt} < 0$

$\Rightarrow$ Temperature of the body is a decreasing function of time and the body is cooling.



Hot Rock in Water

If $T < A,\ then\ \frac{dT}{dt} > 0$

$\Rightarrow$ Temperature of the body is an increasing function of time and the body is heating.

Remark:

1. The physical law is translated into a differential equation
2. If value of k and A are known, we can determine the temperature T of the body at any time t.

Example 15

A cake is taken out from an oven when its temperature has reached 185°F and is placed on a table in a room whose temperature is 75°F. If the temperature of the cake reaches 150°F after half an hour, what will be its temperature after 45 minutes?

Solution: Let T be the temperature of the cake after t minutes.

By Newton's Law of cooling

$\frac{dT}{dt} = -k(T - 75^o)$, where k is the constant of proportionality

$\Rightarrow \int \frac{dT}{T-75} = -k \int 1dt$

$\Rightarrow \log(T - 75) = -kt + c$

$\Rightarrow T - 75 = e^{-kt+c}$

$\Rightarrow T - 75 = e^c e^{-kt}$

$\Rightarrow T - 75 = \lambda e^{-kt}$ ... (1) (Where $e^c = \lambda$)

Substituting $t = 0,\ T = 185^0$ in equation (1), we get

$185 - 75 = \lambda \Rightarrow \lambda = 110$

So, $T - 75 = 110\ e^{-kt}$ ... (2)

Substituting, $t = 30,\ T = 150^0$ in equation (2), we get

$150 - 75 = 110\ e^{-30k}$

$\Rightarrow 75 = 110e^{-30k}$

$\Rightarrow e^{-30k} = \frac{75}{110} = 0.6818$ ... (3)

Substituting, $t = 45$ in equation (2), we get

$T - 75 = 110\ e^{-45k}$

$\Rightarrow T = 75 + 110(e^{-30k})^{1.5}$

$\Rightarrow T = 75 + 110(0.6818)^{1.5}$ (Using (3))

$\Rightarrow T = 136.92$

Hence the temperature after 45 minutes is $137°F$ (approx.)

3.12.6 Carbon Dating

Carbon 14, also known as radiocarbon, is radioactive form of carbon that is found in all living plants and animals. The radiocarbon disintegrates after the plant or animal dies.

Scientists can find an estimate of age of the remains of plants and animals by comparing the amount of radiocarbon in it with those in living plants or animals. This technique is called carbon dating. Cearbon-14 decays exponentially with a half-life of approximately 5700 years, meaning that after 5700 years a given amount of carbon-14 will decay to half of its original amount.

Let $A(t)$ be the mass of carbon-14 after t years

$$\frac{dA}{dt} = -kA$$

$$\Rightarrow \int \frac{1}{A} dA = -k \int 1\, dt$$

$$\Rightarrow \log A = -kt + c$$

$$\Rightarrow A = e^{-kt+c}$$

$$\Rightarrow A = A_0\, e^{-kt} \qquad (\because A = A_0 \; when \; t = 0)$$

Example 16

The amount of radiocarbon present after t years is given by

$A = A_0 e^{-(\ln 2)\left(\frac{1}{5700}\right)t}$, where A_0 is the amount present in the living plants and animals.

a) Find the half-life of radiocarbon.

b) Charcoal from an ancient pit contained $\frac{1}{4}$ of the carbon-14 found in living sample of same size. Estimate the age of the charcoal.

Solution:

a) $\frac{A_0}{2} = A_0\, e^{\left(-\frac{ln2}{5700}\right)t}$

$$\Rightarrow \left(\frac{-ln2}{5700}\right)t = \ln\left(\frac{1}{2}\right)$$

$$\Rightarrow t = \frac{-5700}{ln2}(-ln2) = 5700 \; years$$

∴ The half-life is 5700 years.

b) $\frac{1}{4}A_0 = A_0 e^{\left(-\frac{ln2}{5700}\right)t}$

$$\Rightarrow -\frac{ln2}{5700}t = ln\left(\frac{1}{4}\right)$$

$\Rightarrow -(ln2)t = 5700(ln1 - ln4)$

$\Rightarrow -(ln2)t = -2(ln2)5700$

$t = 11{,}400$

The charcoal is about 11,400 year old

3.12.7 Drug Assimilation into the blood

We readily take pills for diseases like common cold, headaches etc. without having a good understanding of how these medicines are absorbed into the blood or for how long these have effects on our body.

We study how these medicines are absorbed and extracted into the blood stream at different rates.

Case of single common cold pill consumed

Procedure: When a pill is taken, it first dissolves into gastrointestinal tract (GI-tract) and each ingredient is diffused into the blood. These are carried to different body parts on which they act and are removed from the blood stream by kidneys and liver with different rates.

Intake

When a single pill is taken and no more drugs are taken later.

Rate of change of drug in GI-tract = (Rate of drug intake) - (Rate at which drug leaves the GI Tract)

Let $x(t) =$ amount of drug at time t in GI-tract.

And $x(0) = x_0$ = amount of drug taken initially

$\therefore \frac{dx}{dt} =$ Rate of change of drug in GI-tract

$\frac{dx}{dt} = 0 - k_1 x,$ where $k_1 =$ rate at which drug leaves the GI-tract

$\int \frac{dx}{x} = -k_1 \int 1\, dt$

$\Rightarrow \log x = -k_1 t + c$

$\Rightarrow x = e^{-k_1 t + c}$

$\Rightarrow x = \lambda e^{-k_1 t}$ ...(1)

Substituting $t = 0,\ x = x_0$ in (1), we get

$x_0 = \lambda e^{o} \Rightarrow \lambda = x_0$

$\therefore x = x_0 e^{-k_1 t}$,

where $x = x(t)$ is a function of time and gives the amount of drug present in the blood stream at the time t.

Example 17

Nembutal, a sodium salt (sodium pentobarbital) acts as a sedative and has many applications. Suppose Nembutal is used to anesthetize a dog. The dog is anesthetized when its blood stream concentration contains at least 45mg of sodium pentobarbital per kg of the dog's body weight. If the rate of change of sodium pentobarbital say, x in the body, is proportional to the amount of drug present in the body. Show that sodium pentobarbital is eliminated exponentially from the dog's blood stream given that its half-life is 5 hours. What single dose should be administered in order to anesthetize a 50 Kg dog for 1 hour?


Anesthetized dog

Solution: let x be the amount of drug at time t

$\frac{dx}{dt} = -kx,$ where k is the rate at which the drug leaves the blood stream.

$\Rightarrow \int \frac{dx}{x} = -k \int 1\, dt$

$\Rightarrow \log x = -kt + c \Rightarrow \quad x = e^{-kt+c}$

$\Rightarrow x = \lambda e^{-kt}$, where $\lambda = e^{c}$

$\Rightarrow x = x_0 e^{-kt} \quad \dots (1)$ wherex_0 =intial amount of drug

Since, half-life of drug=5 hours

$$\therefore \frac{x_0}{2} = x_0 e^{-5k} \Rightarrow e^{-5k} = \frac{1}{2} \Rightarrow e^{5k} = 2$$

For a dog that weight 50 kg the amount of drug in the body after 1 hour =(45mg/kg) x 50kg=2250 mg

From (1) 2250=$x_0 e^{-k}$ (as $t = 1$)

$\Rightarrow x_0 = 2250(e^{k}) = 2250 \times 2^{1/5} = 2585mg(approx)$

So a single dose of 2585 mg should be administered to anesthetize a 50kg dog for 1 hour.

Exercise 5

1. Find an exponential growth model, $y = y_0 e^{kt}$ that satisfies the stated conditions:
    i. y_0 = 1 and doubling time t = 5 years.
    ii. y(0) = 5 and growth rate = 2%
    iii. y(1) = 1 and y(10) = 100
2. Gaurav deposited ₹ 5000 in an account paying 3% interest compounded continuously for 5 years.
    i. Find the total amount at the end of 5 years.
    ii. How long will it take for the money to double?
3. In a certain culture of bacteria, the number of bacteria increased 5 times in 10 hours. How long did it take for the number of bacteria to double?
4. The amount of oil pumped from one of the wells decreases at the continuous rate of 10% per year. When will the wells output fall to one-fourth of its present value?
5. A cup of tea with temperature 95°C is placed in a room with a constant temperature of 21°C. How many minutes will it take to reach a temperature of 51°C if it cools to 85°C in 1 minute.
6. A cake is removed from an oven at 250°F and left to cool at room temperature which is 70°F. After 30minutes the temperature of the cake is 150°F. After how much time will it be 100°F?
7. Radium decomposes at a rate proportional to the amount present. If half the original amount disappears in 1600 years, find the percentage lost in 100 years.
8. Half-life of radioactive carbon-14 is 5700 years. A certain bone was observed to contain 75% of carbon-14 as compared to what is present in the leaving creatures. Determine its antiquity.
9. If 600 grams of a radioactive substance are present initially and 3 years later only 300 grams remain. How much of the substance will be present after 6 years?
10. The space vehicles are supplied power from nuclear energy derived from radioactive isotopes. The output of the radioactive power supply for a certain satellite is given by the function, $y = 50e^{-0.004t}$ Where y is in watts and t is the time in days.
    a) How much power will be available at the end of 90 days?
    b) How long will it take for the amount of power to be half of its original strength?
11. Use the exponential growth model to show that the time it takes for a population to double (i.e., from an initial number A to 2A) is given by $t = \frac{ln2}{k}$

Answers

Exercise 1

1. order 1, degree 1
2. order 1, degree 1
3. order 2, degree 1
4. order 2, degree 1
5. order 3, degree 2

Exercise 2

7. k = 2

Exercise 3

1. $yy_1 = x$
2. $x + yy_1 = 0$

3. $(x^2 - y^2)y_1 - 2xy = 0$

4. $y_2 - 4y_1 + 4y = 0$

5. $2xy\frac{dy}{dx} - y^2 = 0$

6. $x(yy_2 + y_1^2) = yy_1$

Exercise 4

1. $ln|y| = e^x + x + c$
2. $x^4 + y^4 = c\,x^4 y^4$
3. $x^2 + y^2 + 2x - 4y + c = 0$
4. $e^y + e^{-y} + \frac{x^2}{2} - ln|x| = c$
5. $xe^x - e^x = \sqrt{1 - y^2} + c$
6. $y - x + 2 = 2\,ln|x(y + 2)|$
7. $ln|y| = 2x - 2\,ln|x + 1| + 3\,ln3 - 4$
8. $\frac{e^{3x}}{3} + \frac{e^{-4y}}{4} = \frac{7}{12}$

Exercise 5

1. $(i)\ y = e^{0.1386t}$, $(ii)\ y = 5\,e^{0.02t}$, $(iii)\ y = 0.599e^{0.51t}$
2. (i) Rs 5809 (app), (ii)23.1 years
3. 4.3 hours
4. 13.8 years
5. 6 minutes (app)
6. 1 hour 6 minutes
7. 4.2% (app)
8. 2365.8 years
9. 150 g
10. a) 34.88 watts b)173 days

Online resources

Differential Equations

1. http://www.differentialequationsbook.com/wp-content/uploads/2016/09/SamplePages.pdf
2. https://www.slideshare.net/mdmosharofhosan/differential-equation-64060996?qid=dbdb2c4c-f9bf-453d-9149-7e7b49e463e2&v=&b=&from_search=16

Mathematical Modelling

1. https://www.hec.ca/en/cams/help/topics/Mathematical_modelling.pdf
2. https://application.wiley-vch.de/books/sample/3527407588_c01.pdf
3. https://jvanderw.une.edu.au/Lecture1_IntroToMathModelling.pdf

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